Do paths have the most zero forcing sets?

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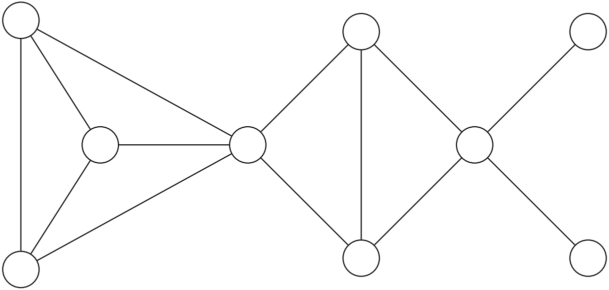
Based on joint work with Anurag Singh

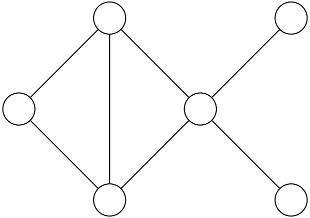
Background

Graphs

Simple, undirected graphs



Zero forcing



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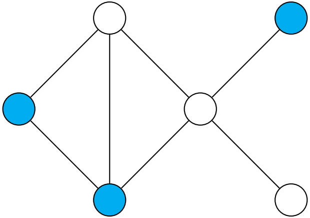

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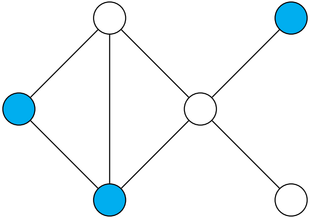
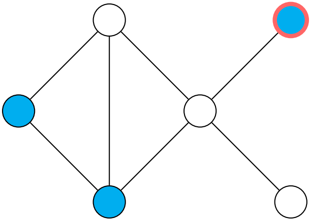


Rule: A blue vertex's *unique uncolored neighbor* can be colored.

Zero forcing



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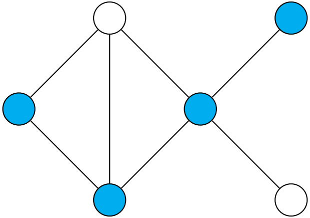

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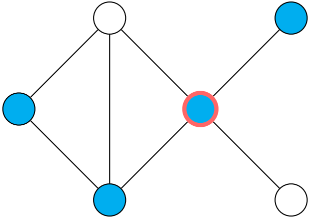

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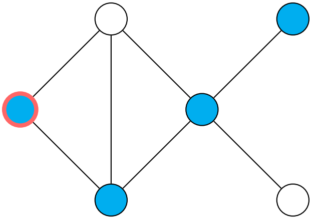

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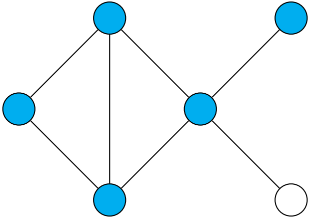

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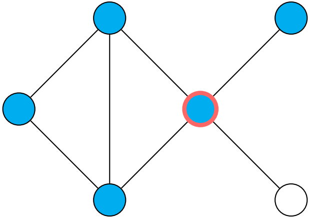

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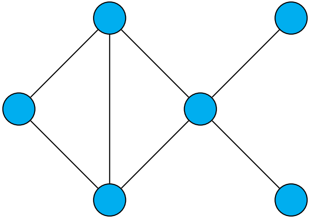
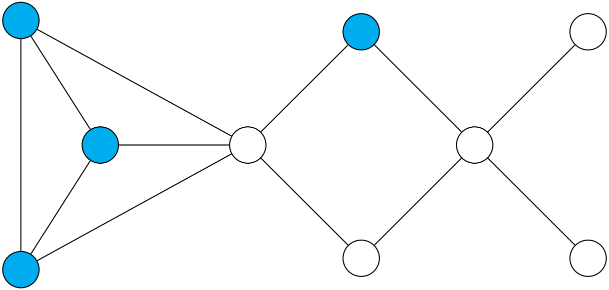


Rule: A blue vertex's *unique uncolored neighbor* can be colored.

Example 2



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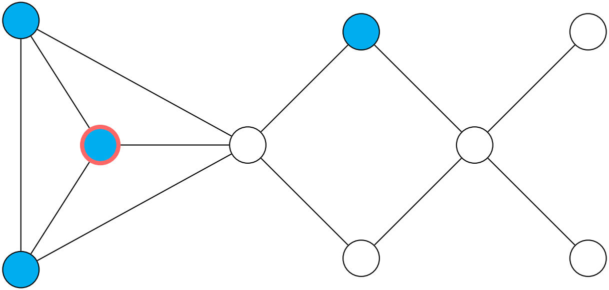

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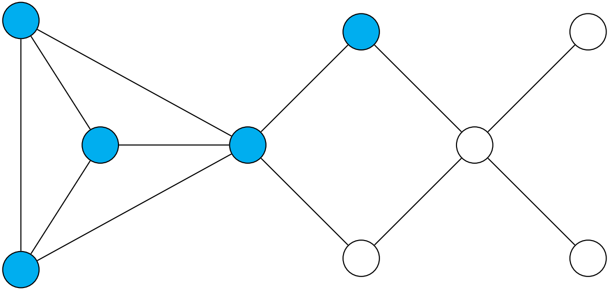

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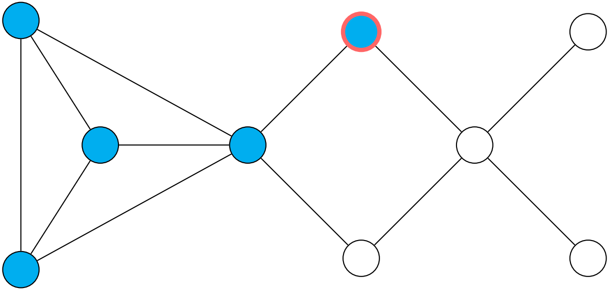

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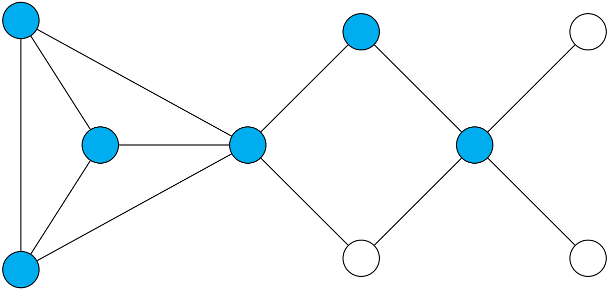

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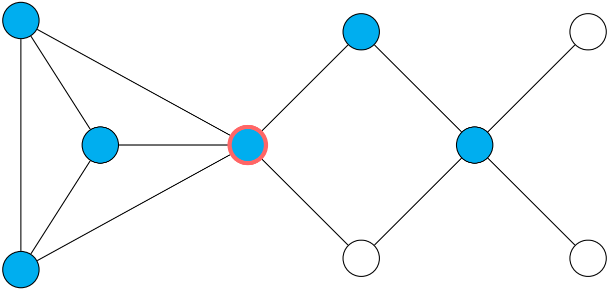

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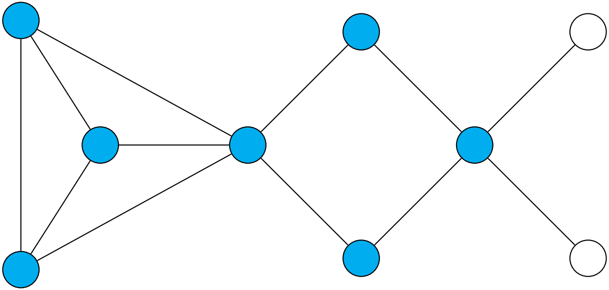


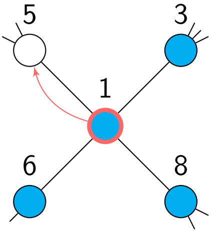
Zero forcing

- A coloring process on graphs
- Start with some vertices colored blue.
- If a blue vertex v has a *unique uncolored neighbor* w, then w can be colored blue.

Definition

A zero forcing set is a set of initially colored vertices capable of coloring the entire graph.

- *'Zero forcing'* since it is used to bound nullity of certain matrices.



$$a_1x_1 + a_3x_3 + a_5x_5 + a_6x_6 + a_8x_8 = 0$$

$$x_1, x_3, x_6, x_8 = 0 \Rightarrow x_5 = 0$$

- Also used to model rumor spreading.

Main focus

Numbers we study

Definition

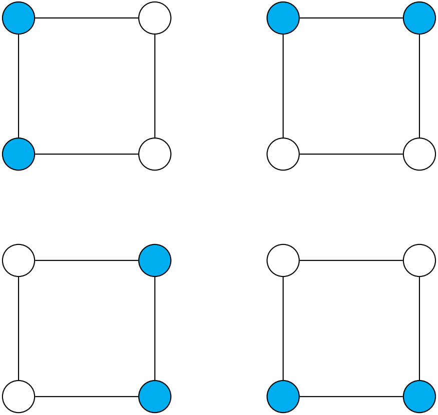
For a graph G and $i \geq 1$, set

$$z(G;i) = \#\{A \in \binom{V(G)}{i} \mid A \text{ is zero forcing}\}.$$

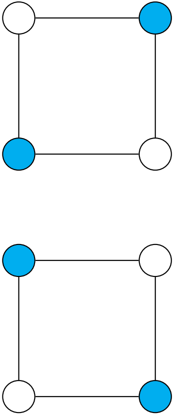
Boyer et al., *The zero forcing polynomial of a graph*, Discrete Appl. Math. (2019).

$z(C_4; 2) = 4$

Forcing

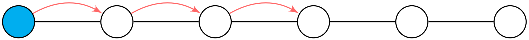


Non-forcing

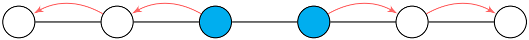


Example: Paths

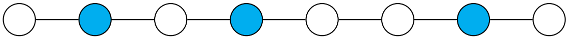
- Contains an end-vertex



- Contains two adjacent vertices



Example: Paths



Proposition (Boyer et al., 2019)

A set of vertices in P_n is forcing if and only if it contains an end-vertex, or two adjacent vertices. Hence, for any $i \geq 1$,

$$z(P_n; i) = \binom{n}{i} - \binom{n-i-1}{i}.$$

Question by Boyer et al.

Conjecture

For any graph G on n vertices,

$$z(G; i) \leq z(P_n; i) \text{ for all } i \geq 1.$$

Our contribution

- Perform operation: $G \to G'$ such that

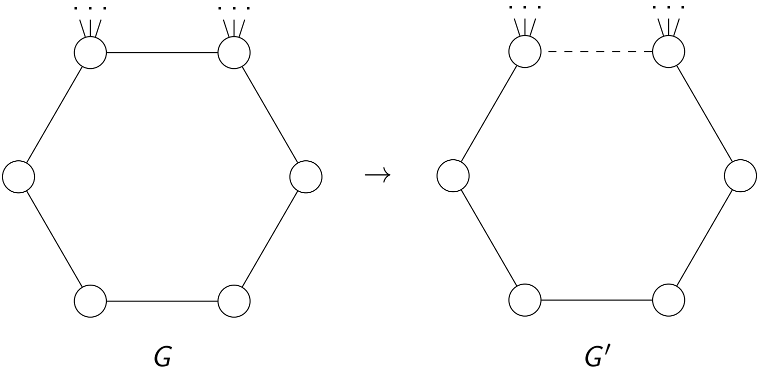
$$z(G;i) \leq z(G';i) \text{ for all } i \geq 1.$$

- Perform several such operations to obtain a path.

$$z(G;i) \leq z(G';i) \leq z(G'';i) \leq \cdots \leq z(P_n;i)$$

Hanging cycle

Delete an edge from which a cycle hangs



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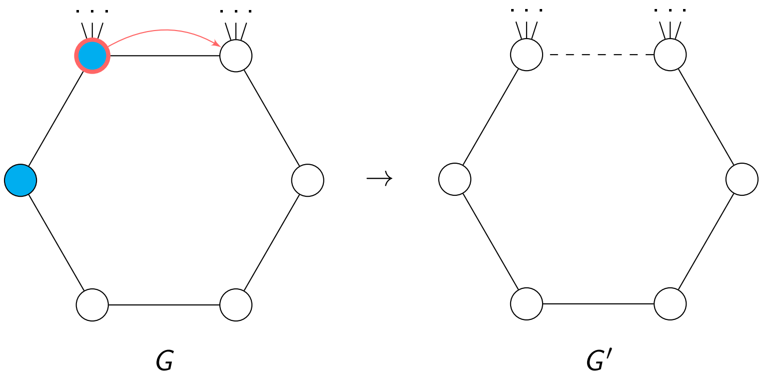

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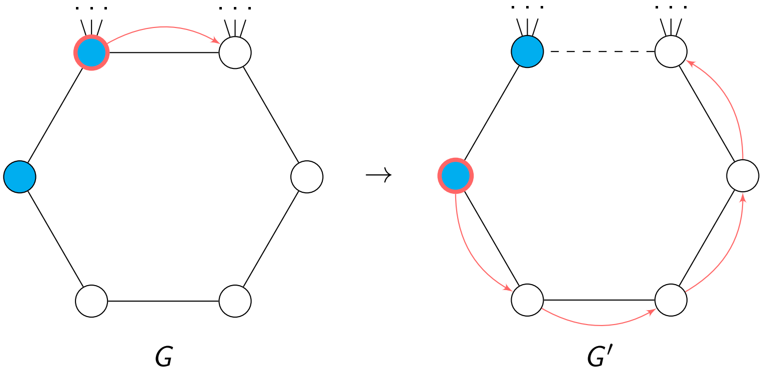
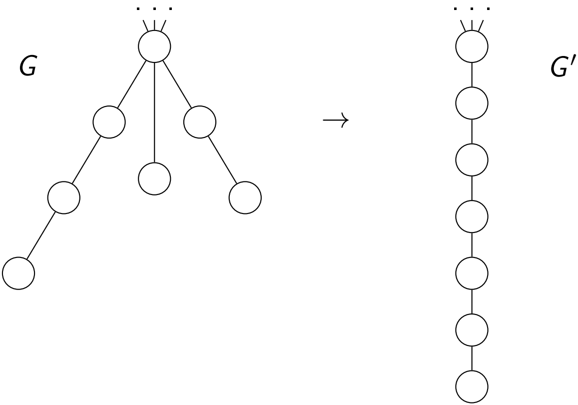
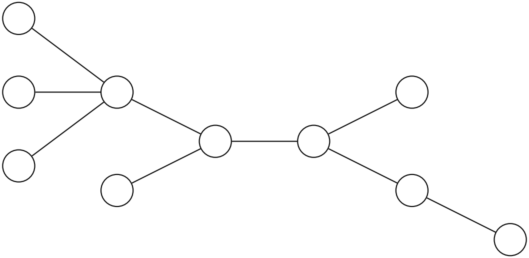


Hanging paths

Combine paths hanging from a vertex



Trees



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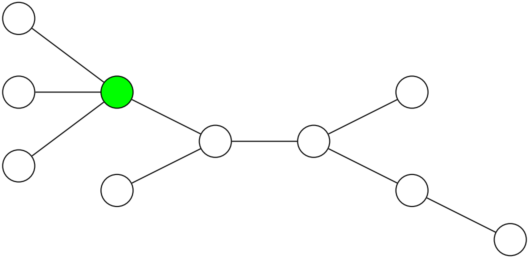

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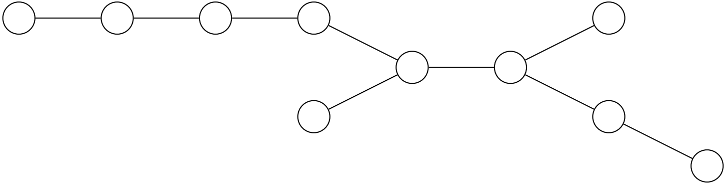

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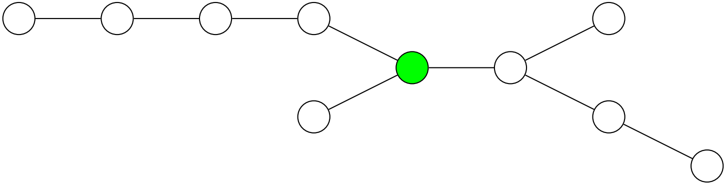

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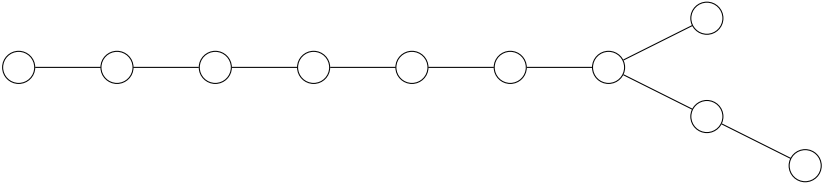

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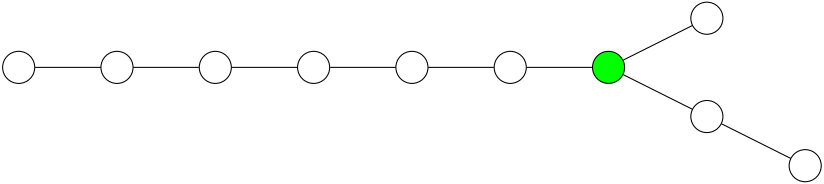

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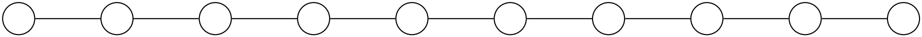


Trees

Theorem (M., Singh, 2025)

For any tree T with n vertices,

$$z(T;i) \leq z(P_n;i) \text{ for all } i \geq 1$$

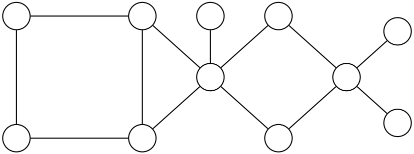
where the inequality is strict for $i < \frac{n}{2}$.

Sam Spiro suggested this case of the conjecture and the strictness.

Other operations

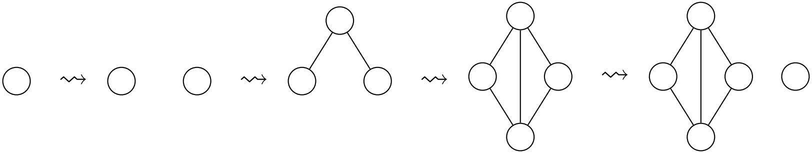
- Wedges of certain types of graphs.
- Taking the cone over a graph.
- Deleting some edges among neighbors of a simplicial vertex.
- Deleting leaves in a graph.

Other results



Theorem (M., Singh, 2025)

All outerplanar graphs, threshold graphs, as well as wedges of such graphs satisfy the conjecture.



Closing remarks

Spanning trees

Done: Any tree T satisfies the conjecture.

Hope: Any graph G will have a spanning tree T with $z(G;i) \leq z(T;i)$.

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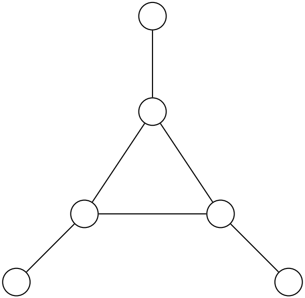
Reality: Not so kind



Equivalence on graphs

Equivalence on graphs with n vertices:

$$[G] = \{H \mid z(G;i) = z(H;i) \text{ for all } i \geq 1\}.$$

What do these equivalence classes look like?

Boyer et al. characterized $[P_n], [K_n], [C_n]$.

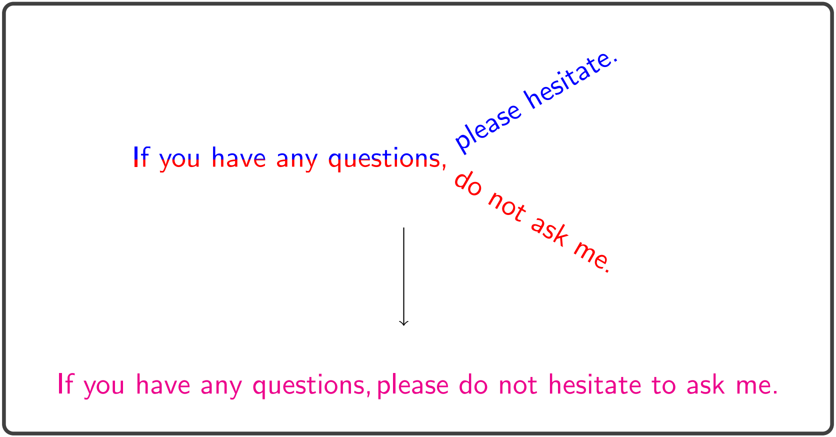
Zero forcing poset

Poset on equivalences classes with

$$[G] \leq [G'] \iff z(G;i) \leq z(G';i) \text{ for all } i \geq 1.$$

- Conjecture is that $[P_n]$ is the maximum element.
- Natural ways to go up from $[G]$ to $[P_n]$ in the poset?
- What else? Cover relations, coatoms etc.

Thank you!



Example: It's good to combine hanging paths.

References

- AIM Special Work Group. *Zero forcing sets and the minimum rank of graphs*. Linear Algebra Appl. (2008).
- K. Boyer, B. Brimkov, S. English, D. Ferrero, A. Keller, R. Kirsch, M. Phillips, and C. Reinhart. *The zero forcing polynomial of a graph*. Discrete Appl. Math. (2019).
- B. Curtis, L. Gan, J. Haddock, R. Lawrence, and S. Spiro. *Zero forcing with random sets*. Discrete Math. (2024).
- K. Menon and A. Singh. *Exploring the influence of graph operations on zero forcing sets*. Discrete Math. (2025).